Easy Finite-Element Implementation of Circuit-Field Problems

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The nodal method of electric circuit theory is fully exploited in our work to provide general building blocks that lead to the non-linear system of equations that model 2D low-frequency electromagnetic problems coupled to electric circuits. The building blocks of the electromagnetic region and circuit system (with arbitrary topology) are easily assembled together using the simple rules of the nodal method. This methodology is restricted to current-forced electric configurations and to a special class of circuit elements. Nevertheless, it is posible to overcome these limitations using the Modified Nodal Analysis method, providing a general methodology that copes with the 2D electromagnetic coupled problem. It is shown that Galerkin or variational approaches are compatible with the building-block approach through winding vectors. This way, a complete and practical computer implementation can be readily written/coded that performs efficiently in terms of computational times and resources. Examples are solved using our computer programs and their results compared with those of validated commercial software.

***Index Terms*—Modified Nodal Analysis, Finite Element Analysis, Low Frequency, Electromagnetic Devices.**

I. Introduction

FINITE ELEMENT (FE) Analysis is a well established technique for modelling low-frequency magnetic devices. Its theory and applications can be found in textbooks [1]–[5], providing an understanding of the physics and mathematics involved. These books include demonstrative computer codes to solve simple magnetostatic and electrostatic problems. Some books even cover problems where low-frequency electromagnetic devices are coupled to circuit systems [2]–[4]. The so-called circuit-field problem is more general since it can deal with problems where (massive or filamentary) conductor currents are unknown and with the electric interaction with external devices represented by equivalent circuits. However, the translation of the theory presented in books [2]–[4] and articles [6] to computer programs is not an easy task.

Electrical and electronic engineers are well familiarised with the nodal method for solution of circuit networks. The nodal approach is frequently translated to a set of rules that can be easily coded into a computer program. The nodal method can be more formally presented with the aid of building blocks [7], which are no more than stamps of the more general Modified Nodal Analysis (MNA) [8], [9]. Once the stamps have been established for each circuit element, a consistent system of equations is obtained by adding each building block to the global matrix and forcing vector. Hence, the global system is easily constructed and solved.

The FE method also leads to elemental matrices and vectors that can be seen as stamps, since they are added to a global system in the same way as the stamps of the MNA. Although the theory behind each stamp is very important, its main task, once known, is to ease writing of computer programs. Indeed, computers do not know about Kirchhoff's current law or variational methods, they only add element contributions to specific locations. Our work aims to provide the required stamps and steps to easily write a computer program from scratch once the theory has been understood.

II. The low-frequency electromagnetic problem

The sort of problems that will be considered in this work obey the following nonlinear equation:

$$\nabla \cdot \nu \nabla A_z = -J_f + \sigma(\frac{\partial A_z}{\partial t} + \nabla V) \quad (1)$$

This equation is valid for 2D Cartesian problems and the definition of each variable, as well as, applicable boundary conditions can be found elsewhere [1]–[5]. Eq. (1) can be coupled to circuit systems through the voltage equation of each conductor. This leads to:

$$\nabla \cdot \nu \nabla A_z - \sigma(\frac{\partial A_z}{\partial t} - \frac{\partial}{\partial t}\frac{\int A_z d\Omega}{\Delta_s}) + \frac{i_m}{\Delta_s} + \frac{n_f i_f}{\Delta_f} = 0 \quad (2)$$

where i_m and i_f are the total currents in the massive and filamentary conductors while Δ_m and Δ_f represent the conductor cross sections. A filamentary region can contain n_f turns. We will consider in this work time-harmonic operation ($\frac{\partial}{\partial t} = j\omega$) and first-order finite elements for the sake of simplicity, but transient problems and higher order element can be readily accommodated.

III. Circuit System

A circuit network of arbitrary topology can be considered with an arbitrary number of circuit elements such as resistances, inductances, capacitances, transformers, voltage and current sources (controlled or independent), and mutually coupled inductances. In addition, FE conductors can be connected to any pair of nodes, allowing circuit interaction between the FE model and the circuit system. The global system of equation is built using the MNA [8], [9].

IV. FE AND CIRCUIT STAMPS

Stamps for Eq. (2) can be found by applying a variational or weighting residual approach [1]–[5]. The stamps of circuit elements are established through the MNA [8].

A. FE arrays

The stamps for an arbitrary first-order FE element, with vertex i, j and k, are:

$$\frac{\nu}{4\Delta_e}\begin{bmatrix} S_{ii} & S_{ij} & S_{ik} \\ S_{ji} & S_{jj} & S_{jk} \\ S_{ki} & S_{kj} & S_{kk} \end{bmatrix}, \qquad \frac{j\omega\sigma\Delta_e}{6}\begin{bmatrix} 1 & 0.5 & 0.5 \\ 0.5 & 1 & 0.5 \\ 0.5 & 0.5 & 1 \end{bmatrix},$$

$$\frac{n_f}{3}\frac{\Delta_e}{\Delta_f}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \frac{1}{3}\frac{\Delta_e}{\Delta_m}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \frac{\sigma_e}{9}\frac{\Delta_e^2}{\Delta_m}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

where $S_{ii} = b_i^2 + c_i^2$, $S_{ij} = b_j b_i + c_j c_i$, $b_i = y_j - y_k$ and $c_i = x_k - x_j$. x_j and x_k are the coordinates of the element vertexes. Other elements are found by permutation of subindices. Δ_e defines the element area. Each stamp corresponds respectively to each term of (2).

B. Circuit Stamps

The stamps of the most common circuit elements are:

$$\begin{bmatrix} y_a & -y_a \\ -y_a & y_a \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -i_s \\ i_s \end{bmatrix}$$

for an addmitance y_a and a current source i_s, respectively. The admittance stamp has a conductance and an inductive or capacitive susceptance, meaning that resistances, inductances and capacitances are readily accommodated. A voltage source e_s with internal impedance z_s contributes with two stamps:

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & -1 & z_s \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 0 \\ -e_s \end{bmatrix}$$

V. CIRCUIT-FIELD COUPLED PROBLEM

The nine stamps just shown allow the solution of a wide class of low-frequency electromagnetic problems. Conventional assembly of FE and circuit stamps is obtained through sequential numbering of nodal vertices and circuit nodes. The coupling is achieved using the terminals of massive and filamentary conductors as circuit nodes and treating them as nonnatural elements of the MNA. This leads to a symmetric equation that is easily constructed:

$$\begin{bmatrix} [K] & & & [N] \\ & [Y] & [A] & [L]^T \\ & [A] & [Z] & \\ [N]^T & [L] & & \end{bmatrix}\begin{bmatrix} [a] \\ [v] \\ [i_{MNA}] \\ [i_{FE}] \end{bmatrix} = \begin{bmatrix} [f_i] \\ [i] \\ [f_v] \\ [0] \end{bmatrix} \qquad (3)$$

where $[K]$ and $[N]$ are FE matrices while $[f_i]$ is a FE forcing vector. $[Y]$ $[Z]$ and $[A]$ are MNA matrices. $[f_v]$ and $[i]$ are MNA forcing vectors. $[L]$ are coupling vectors of FE conductors. $[a]$, $[v]$, $[i_{MNA}]$ and $[i_{FE}]$ are vectors of unknowns. Hence, a functional FE program code can be developed from scratch, which can solve a broad class of problems.

VI. EXAMPLE

Several examples will be shown in the full version of our manuscript. Here we show the simulations results of an induction motor that is fed from a voltage source through series connected impedances. Rotor end rings are represented with an intricate arrangement of circuit elements that couple to the rotor bars. Fig. 1 shows the results (labeled FLD) of the developed computer program using stamps. It also shows the results obtained with commercial software (Flux2D [10]), demonstrating that our approach is not only easy to implement but it is also accurate.

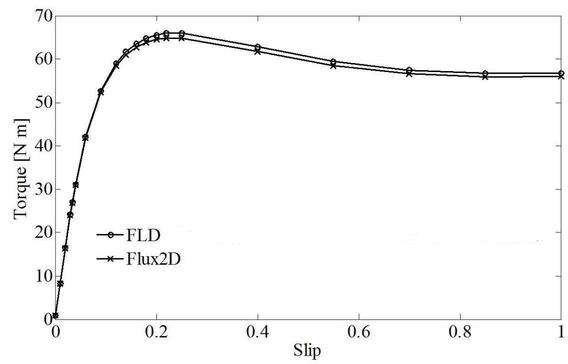

Fig. 1. Torque-speed characteristic of an induction machine

VII. CONCLUSIONS

The concept of stamps is exploited to rapidly and easily write computer programs from scratch to deal with the 2D low-frequency electromagnetic problem. Nine simple stamps have been explicitly provided to solve a wide range of practical problems. This approach can be readily extended to high-order FE elements using exactly the same methodology. Moreover, electrical engineers should find this methodology appealing, as the nodal and MNA circuit analysis methods are part of their common tools.

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